Homological Algebra Seminar Week 6

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2.3 Left derived functors

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $\mathbf{R\text{-}Fun}(\mathcal{A},\mathcal{B})$ and $\delta\text{-}\mathbf{Hom}(\mathcal{A},\mathcal{B})$ be the category of right exact additive functors and the category of homological δ-functors between $\mathcal{A}$ and $\mathcal{B}$ respectively. There is a functor

$$\begin{aligned} \delta\text{-}\mathbf{Hom}(\mathcal{A},\mathcal{B}) &\to \mathbf{R\text{-}Fun}(\mathcal{A},\mathcal{B}) \\ (T_i)_{i\geq 0} &\mapsto T_0. \end{aligned}$$

We now want to construct a functor that goes in the opposite direction

$$\mathbf{R\text{-}Fun}(\mathcal{A},\mathcal{B}) \to \delta\text{-}\mathbf{Hom}(\mathcal{A},\mathcal{B}),$$

that sends right exact functors to universal homological δ-functors. To this end, we will assume that $\mathcal{A}$ has enough projectives.

Definition 2.18. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and assume that $\mathcal{A}$ has enough projectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive right exact functor and let A be an object in $\mathcal{A}$. Fix a projective resolution $P \to A$ and define

$$L_iF(A) := H_i(FP).$$

For any morphism $\delta : A \to A'$ in $\mathcal{A}$, take projective resolutions $P \to A$ and $P' \to A'$. By the comparison Lemma, there is a lift of δ, that is unique up to homotopy

$$\begin{tikzcd} \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\delta"] & 0 \\ \cdots \arrow[r] & P_0' \arrow[r] & A' \arrow[r] & 0. \end{tikzcd}$$

Hence, by applying F to the whole diagram, we get a lift of $F\delta$ that is also unique up to homotopy. This induces a morphism

$$L_i(\delta) : L_iF(A) \to L_iF(A').$$

This defines a functor called the *i-th left derived functor* of F.

Our next goal is to show that L_iF is indeed a well-defined additive functor, under the same hypotheses.

Lemma 2.19. *Let A be an object in $\mathcal{A}$. The object $L_iF(A)$ is well-defined, up to a canonical isomorphism.*

Proof. We will show that $L_iF(A)$ does not depend on the choice of projective resolution. Let $P \to A$ and $Q \to A$ be projective resolutions. Denote by $L_iF(-)_P$ and $L_iF(-)_Q$ the i-th left derived functor constructed using projective resolutions P and Q respectively. By the comparison Lemma, there are morphisms $f : P \to Q$ and $g : Q \to P$ lifting the identity of A. Moreover, the identity $\mathrm{id}_P : P \to P$ is also a lift of the id_A. The situation is illustrated in the following diagram, where the two squares are commutative:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "f"] \arrow[dd, bend right, "\mathrm{id}_P"'] & A \arrow[d, "\mathrm{id}_A"] \\
Q \arrow[r] \arrow[d, "g"] & A \arrow[d, "\mathrm{id}_A"] \\
P \arrow[r] & A.
\end{tikzcd}$$

The comparison Lemma implies that any two lifts must be homotopic. Hence, we get

$$g \circ f \simeq \mathrm{id}_P,$$

since $g \circ f$ is also a lift of the identity. This implies

$$Fg \circ Ff \simeq F\mathrm{id}_P = \mathrm{id}_{FP}.$$

Thus,

$$L_iF(\mathrm{id}_A)_P \circ L_iF(\mathrm{id}_A)_Q = \mathrm{id}_{L_iF(A)_Q}.$$

A symmetrical argument shows that $L_iF(\mathrm{id}_A)_Q \circ L_iF(\mathrm{id}_A)_P = \mathrm{id}_{L_iF(A)_P}$, which concludes the proof. $\square$

Lemma 2.20. *Let $f : A \to A'$ be a morphism in $\mathcal{A}$. The morphism*

$$L_iF(f) : L_iF(A) \to L_iF(A')$$

is well-defined.

Proof. Take projective resolutions P and P' of A and A' respectively. Since two lifts of f are homotopic they must induce the same morphism on homology. $\square$

Proposition 2.21. *Each L_iF is an additive functor.*

Proof. Let A be an object in $\mathcal{A}$ and let $P \to A$ be a projective resolution. Since id_P is a lift of id_A, we must have

$$L_iF(\mathrm{id}_A) = \mathrm{id}_{L_iF(A)}.$$

Now consider morphisms $f : A \to A'$ and $g : A' \to A''$ and chain maps $\tilde{f}, \tilde{g}$ lifting f and g respectively (for some chosen projective resolutions). Since $\tilde{g} \circ \tilde{f}$ is a lift of $g \circ f$ we can compute

$$L_iF(g \circ f) \cong H_iF(\tilde{g} \circ \tilde{f}) \cong H_iF(\tilde{g}) \circ H_iF(\tilde{f}) \cong L_iF(g) \circ L_iF(f).$$

Now let $f^1, f^2 : A \to A'$ be two morphisms with lifts $\tilde{f^1}$ and $\tilde{f^2}$ respectively. As the sum $\tilde{f^1} + \tilde{f^2}$ is a lift of $f^1 + f^2$, we deduce similarly that

$$L_iF(f^1 + f^2) = L_iF(f^1) + L_iF(f^2). \qquad \square$$

Theorem 2.22. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, where $\mathcal{A}$ has enough projectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive right exact functor. The left derived functors $(L_iF)_{i \geq 0}$ form a homological δ-functor.*

Proof. Let

$$0 \to A' \to A \to A'' \to 0$$

be a short exact sequence in $\mathcal{A}$ and let $P' \to A'$, $P'' \to A''$ be projective resolutions. By the Horseshoe Lemma there is a projective resolution $P \to A$ such that the sequence

$$0 \to P' \to P \to P'' \to 0$$

is split exact. Since F is additive, the sequence

$$0 \to F(P') \to F(P) \to F(P'') \to 0,$$

is also split exact. Taking the long exact sequence in homology we get

$$\cdots \to L_{n+1}F(A') \to L_{n+1}F(A) \to L_{n+1}F(A'') \xrightarrow{\partial} L_nF(A') \to L_nF(A) \to \cdots .$$

This proves the existence of a long exact sequence. Now we need to show the naturality of the connecting morphisms. Consider the diagram

$$\begin{tikzcd}
& 0 \arrow[rr] & & P' \arrow[rr] \arrow[dl, "\epsilon'"'] \arrow[dd, "F'"] & & P \arrow[rr] \arrow[dl, "\epsilon"'] & & P'' \arrow[rr] \arrow[dl, "\epsilon''"'] \arrow[dd, "F''"] & & 0 \\
0 \arrow[rr] & & A' \arrow[rr] \arrow[dd, "f'"] & & A \arrow[rr] \arrow[dd, "f"] & & A'' \arrow[rr] \arrow[dd, "f''"] & & 0 & \\
& 0 \arrow[rr] & & Q' \arrow[rr] \arrow[dl, "\eta'"'] & & Q \arrow[rr] \arrow[dl, "\eta"'] & & Q'' \arrow[rr] \arrow[dl, "\eta''"'] & & 0 \\
0 \arrow[rr] & & B' \arrow[rr, "\iota_B"] & & B \arrow[rr, "\pi_B"] & & B'' \arrow[rr] & & 0, &
\end{tikzcd}$$

where the front of the diagram is given and the two front squares commutes, $\epsilon' : P' \to A'$, $\epsilon'' : P'' \to A''$, $\eta' : Q' \to B'$ and $\eta'' : Q'' \to B''$ are any projective resolutions, $\epsilon : P \to A$, $\eta : Q \to B$ are given by the Horseshoe Lemma and

$F' : P' \to Q'$, $F'' : P'' \to Q''$ are lifts of f' and f'' respectively. We will construct $F : P \to Q$ such that

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] \arrow[d, "F'"] & P \arrow[r] \arrow[d, "F"] & P'' \arrow[r] \arrow[d, "F''"] & 0 \\
0 \arrow[r] & Q' \arrow[r] & Q \arrow[r] & Q'' \arrow[r] & 0
\end{tikzcd}$$

commutes. Remember that $P_n = P'_n \oplus P''_n$ and $Q_n = Q'_n \oplus Q''_n$ for every $n \in \mathbb{N}$ by the Horseshoe Lemma. Hence we can define

$$F_n : P'_n \oplus P''_n \to Q'_n \oplus Q''_n$$

by the matrix

$$\begin{pmatrix} F'_n & \gamma_n \\ 0 & F''_n \end{pmatrix},$$

where $\gamma_n : P''_n \to Q'_n$ will be defined inductively. For F to be a lifting of f we must have $\eta F_0 - f\epsilon = 0$. Denote by λ_P and λ_Q the restrictions of ϵ and η to P''_0 and Q''_0 respectively. Then the equality

$$\iota_B \eta' \gamma_0 = -\lambda_Q f''_0 + f\lambda_P$$

must hold. For more clarity, define $g_0 := -\lambda_Q f''_0 + f\lambda_P$. By diagram chasing, one can see that $\pi_B \circ g_0 = 0$. Hence, since $\mathrm{Hom}(P''_0, -)$ is exact (as P''_0 is projective), there exists a map $\beta : P''_0 \to B'$ such that $\iota_B \circ \beta = g_0$. We can then define γ_0 to be a lift of β

$$\begin{tikzcd}
 & P''_0 \arrow[dl, "\gamma_0"'] \arrow[d, "\beta"] & \\
Q'_0 \arrow[r, "\eta'"] & B' \arrow[r] & 0.
\end{tikzcd}$$

We now want to define γ_n for any $n \in \mathbb{N}$. Assume that there is a $n \in \mathbb{N}$ such that γ_i is defined for every $i \in \{0, \ldots, n-1\}$. For F to be a chain map, we must have the equality

$$\begin{aligned}
dF - Fd &= \left[\begin{pmatrix} d' & \lambda \\ 0 & d'' \end{pmatrix}, \begin{pmatrix} F' & \gamma \\ 0 & F'' \end{pmatrix} \right] \\
&= \begin{pmatrix} d'F' - F'd' & (d'\gamma - \gamma d'' + \lambda F'' - F'\lambda') \\ 0 & d''F'' - F''d'' \end{pmatrix} = 0,
\end{aligned}$$

where $[\cdot, \cdot]$ is the matrix commutator. Since both F' and F'' are chain maps, the only condition that γ_n must satisfy is

$$d'\gamma_n = \gamma_{n-1} d'' - \lambda_n F'_n + F''_{n-1} \lambda_n.$$

Define $g_n := \gamma_{n-1}d'' - \lambda_n F'_n + F'_{n-1}\lambda_n$. Using diagram chasing, one can show that $d'g_n = 0$. Since Q' is exact, g_n factors through a map $\beta : P''_n \to d'(Q'_n)$. Hence we can define γ_n to be a lift of β

$$\begin{tikzcd}
 & P''_n \arrow[dl, "\gamma_0"'] \arrow[d, "\beta"] & \\
Q'_n \arrow[r, "\eta'"] & d'(Q'_n) \arrow[r] & 0.
\end{tikzcd}$$

This concludes the proof. $\square$

Before proving our next result, we will need to briefly discuss the notion of pullback.

Definition 2.23. Let $\mathcal{A}$ be an abelian category and let $f : B \to A$, $g : C \to A$ be morphisms. Consider the morphism

$$\varphi := (f, -g) : B \oplus C \to A,$$

induced by the universal property of the coproduct applied to f and $-g$. The *pullback* of f and g is defined to be the kernel $P = \ker(\varphi)$, with induced morphisms $P \to B$ and $P \to C$ that make the diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & B \arrow[d, "f"] \\
C \arrow[r, "g"] & A
\end{tikzcd}$$

commutes. The key property used in the following proof that will not be discussed here is that if f is an epimorphism, so is the morphism $P \to C$. Similarly, if g is an epimorphism, so is the morphism $P \to B$.

Theorem 2.24. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $(L_i)_{i\geq 0}$ be a homological δ-functor from $\mathcal{A}$ to $\mathcal{B}$ and assume that for all objects A in $\mathcal{A}$ and for all $i > 0$ there is an epimorphism $u : P \to A$ such that $L_i(u) = 0$. Then $(L_i)_{i\geq 0}$ is universal.*

Remark 2.25. In particular, if $\mathcal{A}$ has enough projectives, any left derived functor is universal. One can take P projective with an epimorphism $u : P \to A$. Since $L_i(P) = 0$, the map $L_i(P) \to L_iF(A)$ has to be the zero map.

Proof. Let $(T_i)_{i\geq 0}$ be another homological δ-functor and let $\varphi_0 : T_0 \to L_0$ be a natural transformation. We construct for every $n > 0$ a natural transformation $\varphi_n : T_n \to L_n$ by induction. Suppose that there is some $n \in \mathbb{N}$ such that we constructed $\varphi_i : T_i \to L_i$ for every $i \in \{0, \dots, n-1\}$. Let A be an object in $\mathcal{A}$. By assumption, we can choose a short exact sequence

$$0 \to K \to P \to A \to 0,$$

with $P \to A$ as in the statement. Consider the diagram with exact rows

$$\begin{tikzcd}
 & T_n(A) \arrow[r, "\delta"] & T_{n-1}(K) \arrow[r, "\epsilon"] \arrow[d, "\varphi_{n-1}"] & T_{n-1}(P) \arrow[d, "\varphi_{n-1}"] \\
L_n(P) \arrow[r, "0"] & L_n(A) \arrow[r, hook, "\iota"] & L_{n-1}(K) \arrow[r, "\eta"] & L_{n-1}(P).
\end{tikzcd}$$

Using Freyd-Mitchell, we can define $\varphi_n : T_n(A) \to L_n(A)$ on elements. Let $a \in T_n(A)$. By the commutativity of the right square and exactness of the top row we have

$$\eta \circ \varphi_{n-1} \circ \delta(a) = \varphi_{n-1} \circ \epsilon \circ \delta(a) = 0.$$

By exactness of the bottom row, $\varphi_{n-1} \circ \delta(a)$ is in the image of ι and we can define

$$\varphi_n(a) := \iota^{-1} \circ \varphi_{n-1} \circ \delta(a).$$

Moreover, one can show that this map is the unique map making the above diagram commute. Let us assume for now that φ_n is well-defined (this will make sense later). Let $f : A' \to A$ be a morphism and consider the pullback

$$\begin{tikzcd}
P' \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{P} \arrow[d] \\
 & A' \arrow[d, "f"] \\
P \arrow[r] & A,
\end{tikzcd}$$

where the map $\widetilde{P} \to A'$ is an epimorphism given by the assumption. Notice that the map $P' \to \widetilde{P}$ is an epimorphism as $P \to A$ is an epimorphism. Thus we get a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

As both $(L_i)_{i \geq 0}$ and $(T_i)_{i \geq 0}$ are homological δ-functors, each small quadrilateral commutes in the following diagram:

$$\begin{tikzcd}
T_n(A') \arrow[rrr, "T_n(f)"] \arrow[dr, "\delta"] \arrow[ddd, "\varphi_n(A')"'] & & & T_n(A) \arrow[dl, "\delta"'] \arrow[ddd, "\varphi_n(A)"] \\
 & T_{n-1}(K') \arrow[r] \arrow[d, "\varphi_{n-1}"] & T_{n-1}(K) \arrow[d, "\varphi_{n-1}"] & \\
 & L_{n-1}(K') \arrow[r] & L_{n-1}(K) & \\
L_n(A') \arrow[rrr, "L_n(f)"] \arrow[ur, "\delta"] & & & L_n(A). \arrow[ul, "\delta"']
\end{tikzcd}$$

Thus, one can see by diagram chasing that the following holds

$$\delta \circ L_n(f) \circ \varphi_n(A') = \delta \circ \varphi_n(A) \circ T_n(f).$$

Since $\delta : L_n(A) \to L_{n-1}(K)$ is monic, we can cancel the above on the left to see the naturality of φ_n. By taking $A = A'$ and $f = \mathrm{id}_A$ this also shows that $\varphi_n(A)$ does not depend on the choice of P, which shows that it is well-defined.

We now want to show that φ_n commutes with δ_n. Let

$$0 \to A' \to A \to A'' \to 0$$

be a short exact sequence. Let $P \to A$ and $\widetilde{P} \to A''$ be epimorphisms given by the assumption. Similarly as before, the pullback

$$\begin{tikzcd}
P'' \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \widetilde{P} \arrow[d] \\
P \arrow[r] & A \arrow[r] & A''
\end{tikzcd}$$

yields a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K'' \arrow[r] \arrow[d, "g"] & P'' \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d, "\mathrm{id}_{A''}"] & 0 \\
0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0.
\end{tikzcd}$$

Thus, we get another commutative diagram

$$\begin{tikzcd}
T_n(A'') \arrow[r, "\delta"] \arrow[d, "\varphi_n"] & T_{n-1}(K'') \arrow[r, "T(g)"] \arrow[d, "\varphi_{n-1}"] & T_{n-1}(A') \arrow[d, "\varphi_{n-1}"] \\
L_n(A'') \arrow[r, "\delta"] & L_{n-1}(K'') \arrow[r, "L(g)"] & L_{n-1}(A'),
\end{tikzcd}$$

where the right square commutes by naturality of φ_{n-1} and the left square commutes by construction of φ_n. Since $(T_i)_{i\geq 0}$ and $(L_i)_{i\geq 0}$ are both homological δ-functors and by construction of the above morphism of short exact sequences, the horizontal composites are their respective δ maps. This yields the desired commutative relation between δ and φ. $\square$

2.4 Injective Resolutions

Definition 2.26. An object I in an abelian category $\mathcal{A}$ is *injective* if for any monic $f : A \to B$ and any morphism $\alpha : A \to I$, there exists a morphism $\beta : B \to I$ such that $\alpha = \beta \circ f$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[dl, "\beta"] \\
& I &
\end{tikzcd}$$

The following proposition is immediate and thus, given without proof.

Proposition 2.27. *Let $\mathcal{A}$ be an abelian category and let I be an object in $\mathcal{A}$. Then, I is injective in $\mathcal{A}$ if and only if I is projective in $\mathcal{A}^{\mathrm{op}}$.*

Corollary 2.28. *Let $\mathcal{A}$ be an abelian category and let I be an object in $\mathcal{A}$. Then, I is injective if and only if the functor* $\mathrm{Hom}_{\mathcal{A}}(-, I)$ *is exact.*

From this proposition, one can dualize several definitions and results from the projective context. For example, we say that an abelian category $\mathcal{A}$ has *enough injectives* if for every object A in $\mathcal{A}$ there is a monic $A \to I$ for some injective object I in $\mathcal{A}$. Similarly, there is a version of the comparison Lemma in the injective context. Moreover, one has the following criterion for the category of right R-modules:

Proposition 2.29 (Baer's Criterion)**.** *A right R-module E is injective if and only if for every right ideal $J \subseteq R$, every map $J \to E$ can be extended to a map $R \to E$.*

Proof. The proof is omitted as it is typically done in a commutative algebra course. □

We now wish to show that the category R-**mod** has enough injectives. A first step towards this result is the following important example.

Example 2.30. By Baer's criterion, the abelian group $\mathbb{Q}/\mathbb{Z}$ is injective, see Exercise 6.1.

Lemma 2.31. *Let M be a R-module and let A be an abelian group. The canonical morphism*

$$\tau : \mathrm{Hom}_{\mathbf{Ab}}(M, A) \to \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbf{Ab}}(R, A)),$$

defined for any $m \in M$ by

$$\begin{aligned} \tau f(m) : R &\to A \\ r &\mapsto f(mr), \end{aligned}$$

is an isomorphism.

Proof. The inverse is given by

$$\eta : \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbf{Ab}}(R, A)) \to \mathrm{Hom}_{\mathbf{Ab}}(M, A),$$

where

$$\begin{aligned} \eta(g) : M &\to A \\ m &\mapsto g(m)(1). \end{aligned}$$

It is easy to check that these two maps are mutual inverses. □

Proposition 2.32. *Let $R : \mathcal{B} \to \mathcal{A}$ be an additive functor that is right adjoint to some exact functor. Then, for any injective object I in $\mathcal{B}$, the object $R(I)$ is also injective.*

Proof. Denote by $L : \mathcal{A} \to \mathcal{B}$ the left adjoint to R. It suffices to show that $\mathrm{Hom}_{\mathcal{A}}(-, R(I))$ is exact. Given a monic $f : A \to A'$ in $\mathcal{A}$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(A', R(I)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{A}}(A, R(I)) \arrow[d, "\cong"] \\ \mathrm{Hom}_{\mathcal{B}}(L(A'), I) \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(L(A), I) \end{tikzcd}$$

commutes. Since L is exact and I is injective, the bottom map is surjective. Hence, the top map is also surjective and we are done. $\square$

Corollary 2.33. *If I is an injective abelian group, then* $\mathrm{Hom}_{\mathbf{Ab}}(R, I)$ *is an injective R-module.*

Proof. The functor $\mathrm{Hom}_{\mathbf{Ab}}(R, -)$ is right adjoint to the forgetful functor

$$U : R\text{-}\mathbf{mod} \to \mathbf{Ab}.$$ $\square$

Proposition 2.34. *The category R-**mod** has enough injectives.*

Proof. Let M be a right R-module. Define

$$I(M) := \prod_{\mathrm{Hom}_R(M, I_0)} I_0,$$

where $I_0 = \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{R}, \mathbb{Q}/\mathbb{Z})$ is injective, since $\mathbb{Q}/\mathbb{Z}$ is injective. One can verify that a product of injective objects is injective. Now define the morphism

$$\begin{aligned} \iota : M &\to I(M) \\ m &\mapsto \prod_{\varphi \in \mathrm{Hom}_R(M, I_0)} \varphi(m). \end{aligned}$$

We show that it is an injective morphism. Let $m \in M \setminus \{0\}$ and consider the subgroup generated by m. It satisfies either $\langle m \rangle \cong \mathbb{Z}/n\mathbb{Z}$ for some $n > 1$, or $\langle m \rangle \cong \mathbb{Z}$. In the first case, define a morphism of abelian groups

$$\begin{aligned} \langle m \rangle &\to \mathbb{Q}/\mathbb{Z} \\ m &\mapsto \frac{1}{n}. \end{aligned}$$

By the injectivity of $\mathbb{Q}/\mathbb{Z}$, it extends to a morphism of groups $\gamma : M \to \mathbb{Q}/\mathbb{Z}$. Thus, through the identification $\mathrm{Hom}_{\mathbf{Ab}}(M, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(M, I_0)$, we get an element $\varphi \in \mathrm{Hom}_R(M, I_0)$ such that $\varphi(m)(1) = \gamma(m) \neq 0$. In particular, $\iota(m) \neq 0$. If $\langle m \rangle \cong \mathbb{Z}$, the exact same proof works by extending the morphism

$$\begin{aligned} \langle m \rangle &\to \mathbb{Q}/\mathbb{Z} \\ m &\mapsto \frac{1}{2}. \end{aligned}$$

This concludes the proof. $\square$